Capacitor Mathematics

http://gaussmarkov.net

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1 Summary

This article describes how a capacitor interacts with a resistor in a series circuit based on Ohm's law, Kirchoff's laws, and the capacitor law

$$I(t) = C\frac{dV(t)}{dt}$$

where t is time measured in seconds, $I(t)$ is current measured in amperes, C is the value of the capacitor measured in farads, and $V(t)$ is voltage potential measured in volts.

The results derived are

1. Charging a capacitor from a constant voltage source through a current limiting resistor follows the exponential path

$$V(t) = V_0\left(1-\exp\left(-\frac{1}{RC}(t-t_0)\right)\right).$$

where R is the value of the resistor measured in ohms. Discharging is also exponential:

$$V(t) = V_1\exp\left(-\frac{1}{RC}(t-t_1)\right).$$

This equation is useful for predicting the resistor and the time required to discharge safely filter capacitors in a guitar amplifier. Taken together, these two exponential equations describe the output for a rectangular wave through an RC filter as in a low frequency oscillator (LFO).

2. For audio signals, an AC sine wave is the basic input and, as it turns out, the basic output as well. A sine wave voltage across a capacitor is accompanied by a phase-shifted sine wave current through the capacitor:

$$V(t) = A\sin(2\pi f t) \qquad \Longleftrightarrow \qquad I(t) = 2\pi f AC\sin\left(2\pi f t-\frac{\pi}{2}\right)$$

where A is the amplitude measured in volts and f is the frequency of the source measured in hertz. Therefore, for the special case of a sine wave, a

particular ratio of voltage and current of a capacitor is constant over time:

$$\frac{V(t)}{I\left(t-\frac{1}{4f}\right)} = \frac{1}{2\pi f C}$$

Current must be phase shifted and the ratio depends on frequency f as well as capacitance C. This constant voltage-current ratio is reminiscent of Ohm's law for resistors, $V(t)/I(t) = R$.

3. Putting a capacitor into series with a resistor and applying an AC sine wave $V_S(t) = A\sin(2\pi f t)$ results in the capacitor voltage

$$V_C(t) = \frac{A}{1+(2\pi f RC)^2}\left[-2\pi f RC\cos(2\pi f t) + \sin(2\pi f t)\right]$$

which gives the gain of a passive low pass filter

$$\frac{\|V_C(t)\|}{\|V_S(t)\|} = \frac{1}{\sqrt{1+(2\pi f RC)^2}}.$$

The predicted voltage across the resistor is

$$V_R(t) = \frac{A 2\pi f RC}{1+(2\pi f RC)^2}\left[\cos(2\pi f t) + 2\pi f RC\sin(2\pi f t)\right]$$

which gives the gain of a passive high pass filter

$$\frac{\|V_R(t)\|}{\|V_S(t)\|} = \frac{2\pi f RC}{\sqrt{1+(2\pi f RC)^2}}.$$

Both filters have the cutoff frequency

$$f_c = \frac{1}{2\pi RC}.$$

2 Capacitor Behaviour

The behaviour of a capacitor is described by the equation

$$I(t) = C\frac{dV(t)}{dt}$$

where t is time, $I(t)$ is current through the capacitor at time t, and $V(t)$ is the voltage across the capacitor leads at time t. This equation says that the current through a capacitor is proportional to the rate of change of the voltage across the capacitor. The factor of proportionality is the capacitance C of the capacitor. Time enters the relationship because it involves a rate of change with respect to time.

One way to understand capacitors is to compare them to resistors. The current through a resistor is proportional to the voltage across the resistor,

$$I(t) = \frac{1}{R} V(t).$$

The factor of proportionality is one over the resistance R of the resistor. So if the voltage across a resistor $V(t)$ is not zero and constant, then a constant nonzero current runs through the resistor. Constant voltage corresponds to a rate of change equal to zero,

$$V(t) = V_0 \qquad \Longleftrightarrow \qquad \frac{dV(t)}{dt} = 0.$$

So if the voltage across a capacitor is constant (zero or not), then no current will flow. This is why it is said that "capacitors block DC." (This statement needs to be qualified and we will do that later.)

Another property described by the equation is that capacitor behaviour is not affected by the level of the voltage. Current is linked to changes in voltage and not levels. For a resistor, the current increases with the voltage potential. This is not so for a capacitor. Conversely, a change in voltage across a resistor is reflected immediately and proportionately in a change in current. A change in voltage across a capacitor is accompanied by a current. Typically that current contributes to another change in voltage and another current. A process unfolds that takes time to work itself out.

To get a constant current to flow through a capacitor, the change in voltage must be constant. And this reveals one way the capacitor equation breaks down as a description of real capacitors. If we increase it at a constant rate then voltage grows without bound. Eventually, the voltage differential across a capacitor becomes too large and the capacitor breaks down. I believe it creates a short circuit. The capacitor equation places no bounds on the operating voltage of capacitors and so fails to describe this behaviour.

Something similar happens to resistors. One cannot apply any voltage differential to a resistor. If the power generated becomes large enough, a resistor will not dissipate heat energy sufficiently and will break down.

3 Discharging and Charging

The capacitor equation describes how capacitors behave like chargeable batteries, storing energy in response to a rise in voltage potential across their leads and discharging in response to a drop in voltage. For this reason, capacitors are often said to *resist* voltage changes (by drawing current from or supplying current to the source of the change).

Consider an RC series circuit with initial conditions at 0V. In an instant, the source voltage jumps to V_0 volts. This would happen if we switched from ground to a battery connection. The resistor will react instantaneously and current will

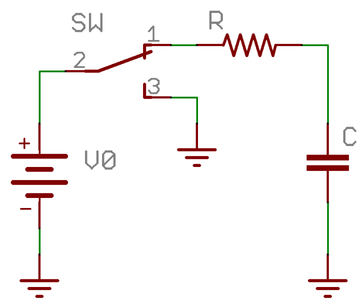

Figure 1: Switched RC Circuit

flow through the resistor according to Ohm's law:

$$I(t) = \frac{V_0 - V_C(t)}{R}$$

Note that the voltage drop across the resistor is determined along with the voltage across the capacitor. If we label the voltage across the capacitor $V_C(t)$, then the voltage drop accompanying the current through the resistor is $V_0 - V_C(t)$, so that is the expression we put into Ohm's law.

Because this is a series circuit, the current through the capacitor is equal to the current through the resistor so that we can put the same $I(t)$ into the capacitor equation:

$$I(t) = C\frac{dV_C(t)}{dt}$$

which tells us how the voltage differential across the capacitor responds.

We can eliminate $I(t)$ from these two equations and solve for voltages to obtain

$$\frac{dV_C(t)}{dt} = \frac{V_0 - V_C(t)}{RC},$$

which is a differential equation for the voltage across the capacitor, $V_C(t)$. Before solving for $V_C(t)$, note that two constants determine the solution, V_0 the initial voltage jump and $\tau = RC$. Many pairs of resistor-capacitor values will give the same predicted behaviour as long as the product of their values is τ. $\tau = RC$ is often called the *time constant* because of this role it plays in RC (resistor-capacitor) circuits.

The solution to the differential equation is

$$V_C(t) = V_0\left[1 - \exp\left(-\frac{1}{RC}(t - t_0)\right)\right].$$

Now you can see why $\tau = RC$ is called the time constant. In this solution, τ determines how time affects the path of the capacitor's voltage. You may also see that the initial voltage change, V_0, only determines the *scale* of that path. The pattern over time is the same for all resistor-capacitor pairs with the same time constant. Finally, take note that the capacitor voltage theoretically never reaches V_0, although it gets closer and closer.

The current associated with this voltage is

$$\begin{aligned} I(t) &= \frac{V_0 - V_C(t)}{R} \\ &= \frac{V_0}{R} \exp\left(-\frac{1}{RC}(t - t_0)\right) \end{aligned}$$

Initially the current equals V_0/R, which is what would flow through the resistor if that were the only component in the circuit. As time passes, the current falls towards zero. According to this description, there is always a current but it becomes very small. Therefore, "capacitors block DC' only in the long run. There can be an initial non-negligible current. Incidentally, this current is the source of the pop heard when switching on some stompbox effects. A *pull-down resistor* remedies this problem.

Differential Equation Solution

Here is a description of a solution to this differential equation. First we separate the voltage terms from the time constant, putting them on different sides of the equation:

$$\frac{\frac{dV_\text{C}(t)}{dt}}{V_0 - V_\text{C}(t)} = \frac{1}{RC}.$$

This is helpful because the right-hand side is a recognizable derivative:

$$\frac{\frac{dV_\text{C}(t)}{dt}}{V_0 - V_\text{C}(t)} = \frac{d}{dt}[-\ln(V_0 - V_\text{C}(t))]$$

So is the left-hand side:

$$\frac{1}{RC} = \frac{d}{dt}\left(\frac{1}{RC}t\right)$$

Because the derivative of a constant is zero, we can only be sure that these two derivatives integrate to equal expressions when we include a constant term, a_0, that will be determined later:

$$-\ln[V_0 - V_\text{C}(t)] = \frac{1}{RC}t + a_0$$

Exponentiating both sides of this equation allows us to solve for $V_\text{C}(t)$:

$$V_\text{C}(t) = V_0 - \exp\left(-\frac{1}{RC}t - a_0\right)$$

Finally, the constant of integration, a_0, is determined because we know $V_C(t)$ at a particular moment in time. Initially, at $t = t_0$, there is no voltage across the capacitor so that

$$\begin{aligned} 0 &= V_{\mathrm{C}}(t_0) \\ &= V_0 - \exp\left(-\frac{1}{RC} t_0 - a_0\right) \end{aligned}$$

Solving this equation for a_0 and substituting the result into $V_{\mathrm{C}}(t)$ gives the final solution:

$$V_{\mathrm{C}}(t) = V_0 \left[1 - \exp\left(-\frac{1}{RC}(t - t_0)\right)\right].$$

Discharging

If we flip the switch in Figure 1 and remove the battery source, the voltage across the capacitor will dissipate. For a resistor, the voltage would immediately drop to zero. But for a capacitor it takes time. The pattern of discharge can be found using similar reasoning. If the voltage across the capacitor is V_1 at $t = t_1$ when the circuit is switched to ground then

$$V_{\mathrm{C}}(t) = V_1 \exp\left(-\frac{1}{RC}(t - t_1)\right)$$

In this circuit, the voltages across the capacitor and the resistor are equal so the current at time t is given by

$$I(t) = \frac{V_1}{R} \exp\left(-\frac{1}{RC}(t - t_1)\right)$$

so that like voltage, current is greatest at the beginning of the discharge.

Alternating between charging and discharging a capacitor is the same as feeding the capacitor a rectangular AC wave. The SPICE simulations on the webpage about capacitors show the exponential functions given here.

4 Discharging a Capacitor with a Resistor

We can apply this equation for capacitor discharge to working on guitar amplifiers. Before you work on the inside of an amp, you should discharge the filter capacitors. These may retain a voltage potential after the amplifier is turned off and unplugged. The potential may be large enough to kill you if your body serves as the route to ground. A widely recommended method for releasing this voltage potential is to connect the leads of a resistor across the leads of one of the filter capacitors. What value resistor should you use? What power rating should the resistor have? How long does it take to discharge the capacitors until it is safe?

We can use the expression above to figure out approximately what happens for different choices. An amp tech gave a friend of mine a big fat 1Ω resistor and told him to use that. It is big and fat because it has a high power rating. Several hundred volts can be stored in filter caps. If V_1 is only 100V then, the initial current will be

$$I(0) = \frac{V_1}{R} = \frac{100\text{V}}{1\Omega} = 100\text{A}$$

which is a lot of current. Voltage times current equals power. In this example 100V times 100A equals 10kW. I wonder how fat that resistor is? This does not sound like a sensible method to me. But you can find recommendations on the internet to use $10W$ resistors and this gives you an idea why that might be.

Suppose instead that you are going to use a resistor with $R = 100\text{K}$. You surely have a 1/4W 100K resistor lying around. Let's see how that will work. First, we see that the initial current is

$$\frac{100\text{V}}{100{,}000\Omega} = 1\text{mA}$$

which is less than the current requirements of a typical stompbox circuit. In this example initial power will be 100V times 1mA equals 0.1W. So a 1/4W resistor will do the job.

How long will it take? Suppose the voltage potential is stored in a $C = 200\mu\text{F}$ capacitor. Let us say that we want to reduce the voltage to 1% of the initial voltage. If 300V are stored then we will reduce the voltage to 3V. In this case, $\tau = 100 \times 10^3\Omega \times 200 \times 10^{-6}F = 20$. Now

$$0.01 = \exp\left(-\frac{1}{20}t\right)$$

which gives

$$t = -20 \times \ln(0.01) \approx 92 \text{ seconds}$$

That means you are going to have to be patient before proceeding to work on your amplifier. You will have to wait one and a half minutes. But think of the time you saved not having to look for a special $10W$ resistor. What the heck? Why not wait two minutes and then measure the voltage across the capacitor when you are done just to be doubly sure? And remember, always keep one hand in your pocket.

5 Alternating Current

The mathematical description of sound and its associated AC voltages is generally carried out using sine and cosine functions. When the voltage applied across a capacitor, $V_\text{C}(t)$, is a sine wave

$$V_\text{C}(t) = A\sin(2\pi f t)$$

where f is the frequency of the AC signal (measured in Hertz or cycles per second) and A is the amplitude of the signal (measured in volts). This means that the rate of change in voltage is

$$\begin{aligned}\frac{dV_{\text{C}}(t)}{dt} &= 2\pi f A\cos(2\pi f t)\\ &= 2\pi f A\sin\left(2\pi f t + \frac{\pi}{2}\right)\end{aligned}$$

which is a phase shifted sine wave with the same frequency f, the amplitude $2\pi f A$ volts per second, and a phase shift of $\pi/2$ radians (or 90 degrees). Using the capacitor equation, we can see that the corresponding current through the capacitor is

$$I(t) = 2\pi f AC\sin\left(2\pi f t + \frac{\pi}{2}\right)$$

so that the current has the same frequency as voltage but the current and the voltage are out of phase by $\pi/2$ radians. To measure the phase shift in cycles, write

$$2\pi f t + \frac{\pi}{2} = 2\pi f\left(t + \frac{1}{4f}\right)$$

to see that this is one quarter of the time for one cycle, $1/f$. In addition, because this one quarter cycle is subtracted from t, current is one-quarter cycle "ahead" of voltage in time.

Using this fact, one can write

$$\frac{V_{\text{C}}(t)}{I\left(t - \frac{1}{4f}\right)} = \frac{1}{2\pi f C}$$

which looks like Ohm's law for resistors except that resistance is replaced by $1/(2\pi f C)$ and current is phase-shifted into alignment with voltage. This analog to resistance is called *reactance*. As the frequency f approaches zero, reactance approaches infinity. This shows another way that "capacitors block DC" because DC is effectively AC with frequency equal to zero and magnitude determined by amplitude and phase. Another expression for this phenomenon, is that "capacitors are like an open circuit" at zero frequency. On the other hand, as f approaches infinity, reactance falls to zero. In other words, at high frequencies a capacitor behaves like a short circuit.

A simpler observation, which ignores the phase shift, is to take the ratio of the amplitudes of voltage and current. If we indicate amplitude with double vertical bars, then we can write

$$\frac{\|V_{\text{C}}(t)\|}{\|I(t)\|} = \frac{A}{2\pi f AC} = \frac{1}{2\pi f C}$$

as a comparable analogy to Ohm's law.

6 RC Filters

Once again, consider a resistor and a capacitor in series to ground, but suppose that the voltage source is delivering a sine wave:

$$V_{\text{S}}(t) = A\sin(2\pi f t)$$

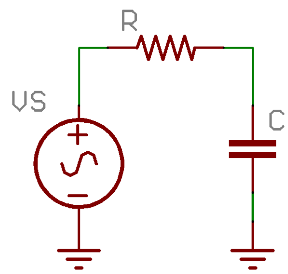

Figure 2: AC Supply to RC Circuit

Kirchoff's voltage law implies that

$$V_S(t) = V_R(t) + V_C(t)$$

where $V_R(t)$ is the voltage across the resistor at time t and $V_C(t)$ is the corresponding voltage across the capacitor. Kirchoff's current law implies that

$$I_R(t) = I_C(t)$$

were $I_R(t)$ and $I_C(t)$ are the currents through the resistor and capacitor respectively. In addition, we have the equations describing resistor and capacitor behaviour:

$$I_R(t) = \frac{1}{R} V_R(t)$$
$$I_C(t) = C \frac{dV_C(t)}{dt}$$

where R is the value of the resistor and C is the value of the capacitor.

As before, one can eliminate current by combining the two behavioural equations to get

$$V_R(t) = RC \frac{dV_C(t)}{dt}$$

and then substitute for $V_R(t)$ and $V_S(t)$ to obtain an equation that contains one unknown, $V_C(t)$:

$$A \sin(2\pi f t) = V_C(t) + RC \frac{dV_C(t)}{dt}$$

There are several popular routes to solving this differential equation. We will use "guess and check." Knowing that neither capacitors nor resistors change the shape or frequency of AC sine waves, one might guess that the answer is a sine wave with the same frequency f, but with an amplitude and phase shift that are part of the solution.

There is also a trigonometric motivation for this guess: two sine waves of the same frequency always sum to another sine wave of that frequency. If $V_C(t)$ is a sine wave with frequency f, then $dV_C(t)/dt$ is phase-shifted sine wave with the same frequency and the left-hand sum in the differential equation will be a third sine wave with the given frequency. That is exactly what is on the right-hand side of the differential equation.

Before checking our guess at the solution, the next section summarizes the background trigonometry.

The Angle Sum Formula

Our guess at the solution is based upon two facts about sine and cosine functions. First, for any angle θ

$$\sin^2\theta + \cos^2\theta = 1.$$

Second, for the sum of any two angles θ and ω

$$\sin(\theta + \omega) = \cos\theta\sin\omega + \sin\theta\cos\omega$$

An important special case is $\varphi = \pi/2$ (or 90°) or a one-quarter cycle phase shift:

$$\begin{aligned}\sin\left(2\pi f t + \frac{\pi}{2}\right) &= \sin\left(\frac{\pi}{2}\right)\cos(2\pi f t) + \cos\left(\frac{\pi}{2}\right)\sin(2\pi f t)\\ &= \cos(2\pi f t)\end{aligned}$$

so that the cosine wave is a sine wave phase-shifted by one-quarter cycle.

Phase

This angle sum formula gives another way to parameterize phase shifts:

$$\begin{aligned}\sin\left(2\pi f t + \varphi\right) &= \cos\varphi\sin(2\pi f t) + \sin\varphi\cos(2\pi f t)\\ &= a\cos(2\pi f t) + b\sin(2\pi f t)\end{aligned}$$

where a and b are determined by φ:

$$a = \sin\varphi \qquad \text{and} \qquad b = \cos\varphi$$

This particular weighted sum of sine and cosine functions is equal to a sine wave with an amplitude equal to one. This is reflected in the property that the sum of squares of the weights a and b equals one:

$$a^2 + b^2 = \cos^2\varphi + \sin^2\varphi = 1.$$

The weights a and b are related to the phase, which we can recover from

$$\frac{\sin\varphi}{\cos\varphi} = \frac{a}{b} \qquad \Longleftrightarrow \qquad \varphi = \arctan\left(\frac{a}{b}\right)$$

If we assign an arbitrary amplitude A to the sine wave then

$$A\sin\left(2\pi f t + \varphi\right) = Aa\cos(2\pi f t) + Ab\sin(2\pi f t)$$

and the coefficients Aa and Ab no longer have a sum of squares equal to one. Instead,

$$(Aa)^2 + (Ab^2) = A^2(a^2 + b^2) = A^2$$

As a result, given a general weighted combination

$$A\sin\left(2\pi f t + \varphi\right) = c\cos(2\pi f t) + d\sin(2\pi f t)$$

for arbitrary c and d, it follows that

$$\begin{aligned} A &= \sqrt{c^2 + d^2}, \\ \varphi &= \arctan\left(\frac{c}{d}\right). \end{aligned}$$

Summing Sine Waves

The angle sum formula also provides a method to derive a sine wave as the sum of two sine waves with the same frequency but different phase:

$$\begin{aligned} \sin(2\pi f t) + \sin(2\pi f t + \omega) &= \sin(2\pi f t) + \cos\omega\sin(2\pi f t) + \sin\omega\cos(2\pi f t) \\ &= \sin\omega\cos(2\pi f t) + (1 + \cos\omega)\sin(2\pi f t) \\ &= A\sin\left(2\pi f t + \varphi\right) \end{aligned}$$

where the amplitude A and the phase φ of the resulting sine wave can be found using the formulas above:

$$\begin{aligned} A &= \sqrt{\sin^2\omega + (1 + \cos\omega)^2} \\ \varphi &= \arctan\left(\frac{1 + \cos\omega}{\sin\omega}\right) \end{aligned}$$

Voltage of the Capacitor: Guess and Check

So let

$$V_C(t) = a\cos(2\pi f t) + b\sin(2\pi f t)$$

be a guess at the solution and check whether we can find coefficients a and b that make this guess satisfy the differential equation. Our guess implies

$$\frac{V_C(t)}{dt} = -2\pi f a\sin(2\pi f t) + 2\pi f b\cos(2\pi f t)$$

and substituting these into the differential equation gives

$$A\sin(2\pi f t) = (a + 2\pi f RCb)\cos(2\pi f t) + (b - 2\pi f RCa)\sin(2\pi f t)$$

By matching coefficients on the sine and cosine functions on each side of the equality, we get two equations for the two unknowns, a and b:

$$\begin{aligned} 0 &= a + (2\pi f RC)b \\ A &= b - (2\pi f RC)a \end{aligned}$$

which we can solve to get

$$\begin{aligned} b &= \frac{A}{1+(2\pi fRC)^2} \\ a &= -2\pi fRC\frac{A}{1+(2\pi fRC)^2} \end{aligned}$$

With these values for a and b, our guess at the solution does actually satisfy the differential equation and we have confirmed our guess. So our solution is

$$V_{\mathrm{C}}(t) = \frac{A}{1+(2\pi fRC)^2}\left[-2\pi fRC\cos(2\pi ft)+\sin(2\pi ft)\right]$$

The amplitude of the solution is

$$\sqrt{a^2+b^2} = \frac{A}{\sqrt{1+(2\pi fRC)^2}}$$

and phase is

$$\arctan\left(\frac{a}{b}\right) = \arctan(-2\pi fRC)$$

You can see why the time constant $\tau = RC$ and angular frequency $\omega = 2\pi f$ are widely used notation. Substituting these definitions into the AC voltage equation makes it easier to read:

$$V_{\mathrm{C}}(t) = \frac{A}{1+(\omega\tau)^2}\left(-\omega\tau\cos(\omega t)+\sin(\omega t)\right)$$

We will adopt this notation from now on.

Current

We can also solve for the current through the circuit:

$$\begin{aligned} I_{\mathrm{C}}(t) &= C\frac{dV_{\mathrm{C}}(t)}{dt} \\ &= \frac{A\omega C}{1+(\omega\tau)^2}\left[\cos(\omega t)+\omega\tau\sin(\omega t)\right] \end{aligned}$$

Voltage of the Resistor

Finally, we can get an expression for the voltage across the resistor, $V_{\mathrm{R}}(t)$:

$$\begin{aligned} V_{\mathrm{R}}(t) &= V_{\mathrm{S}}(t) - V_{\mathrm{C}}(t) \\ &= A\sin(\omega t) - \frac{A}{1+(\omega\tau)^2}\left[-\omega\tau\cos(\omega t)+\sin(\omega t)\right] \\ &= \frac{A\omega\tau}{1+(\omega\tau)^2}\left[\cos(\omega t)+\omega\tau\sin(\omega t)\right] \end{aligned}$$

Notice, as a check on our manipulations, that $I(t)$ and $V_\mathrm{R}(t)$ have proportional coefficients on the sine and cosine functions. This means that they are in phase, just as they must be. In addition, the ratio of this voltage and current equals R, which also must hold.

The amplitude of this AC sine wave is

$$\frac{A\omega\tau}{1+(\omega\tau)^2}\sqrt{1+(\omega\tau)^2} = \frac{A\omega\tau}{\sqrt{1+(\omega\tau)^2}}.$$

The phase is

$$\arctan\left(\frac{1}{\omega\tau}\right).$$

7 High Pass Filter

The passive high pass filter (HPF) takes the voltage across the resistor as the output, as in Figure 3. The *gain* is the ratio of the amplitudes of output voltage and the input voltage:

$$\frac{\|V_\mathrm{R}(t)\|}{\|V_\mathrm{S}(t)\|} = \frac{\omega\tau}{\sqrt{1+(\omega\tau)^2}} \tag{1}$$

The gain is actually a number less than one, so that the passive HPF has a loss of signal no matter what the frequency. As frequency approaches zero, the gain falls to zero. As frequency approaches infinity, the gain rises towards one.

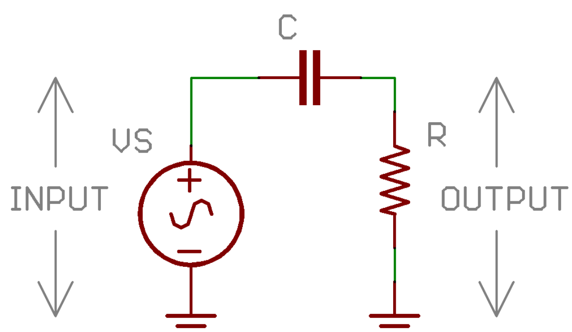

Figure 3: RC High Pass Filter Schematic

Audio gain is usually measured in decibels. To translate between voltage gain and decibels, use the formula

$$\text{gain in decibels} = 20\log_{10}(\text{voltage gain})$$

so that a graph of the HPF gain in decibels as a function of frequency f is usually

$$\text{gain in decibels} = 20\log_{10}\left(\frac{2\pi f RC}{\sqrt{1+(2\pi f RC)^2}}\right).$$

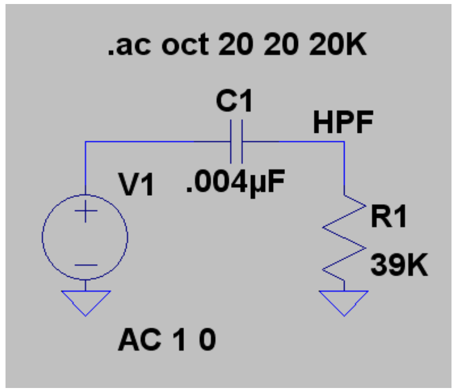

Figure 4: SPICE High Pass Filter AC Analysis

This is exactly what you will see in the AC analysis of a SPICE simulation. For example, part of the tone control of a Big Muff Pi is a passive HPF with a 0.004μF capacitor and a 39K resistor (see a schematic from the layouts section of gauss-markov.net). Using LTSpice/SWCadIII with the schematic in Figure 4 gives the graph in Figure 5. The solid green line is the gain in decibels. The cross-hairs give a reading on a particular point of the graph: −3.02279dB at 1.01751KHz. If we plug these values into the formula, we get

$$\begin{aligned} 2\pi f RC &= 2\pi\left(1.01751\times 10^{3}\right)\left(39\times 10^{3}\right)\left(0.004\times 10^{-6}\right) \\ &\approx 0.997\,339\,805\,578 \end{aligned}$$

and

$$\begin{aligned} 20\log_{10}\left(\frac{0.997\,339\,805\,578}{\sqrt{1+(0.997\,339\,805\,578)^2}}\right) &\approx 20\log_{10} 0.706\,164\,381\,456 \\ &\approx -3.021\,883\,835\,92 \end{aligned}$$

which agrees to a reasonable numerical accuracy.

To interpret such numbers, treat −6dB as halving perceived volume. This −3dB change corresponds to roughly a 25% drop in volume. But note that these guidelines are quite loose. Perceptions vary quite a bit across individuals. It is often claimed that the decibel scale was originally chosen so that a 10dB drop would halve volume.

8 Low Pass Filter

The passive low pass filter (LPF) takes the voltage across the capacitor as the output, as in Figure 6. The gain is the ratio of the amplitudes of output voltage

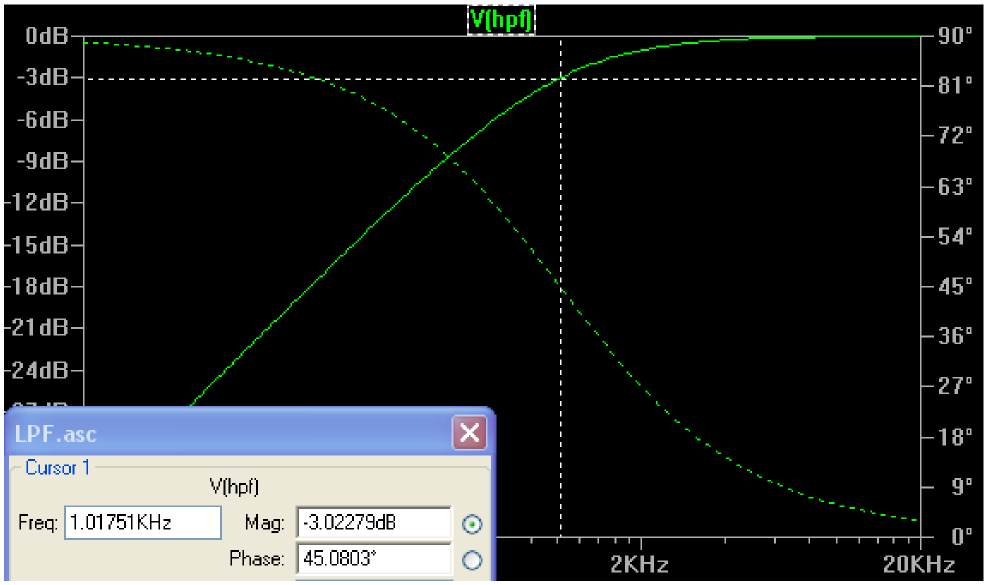

Figure 5: SPICE High Pass Filter AC Analysis

and the input voltage:

$$\frac{\|V_C(t)\|}{\|V_S(t)\|} = \frac{1}{\sqrt{1+(\omega\tau)^2}} \tag{2}$$

This filter gain equation behaves in an opposite fashion to the previous case. As the frequency f of a sine wave input approaches zero, the gain rises to one. As f approaches infinity, the gain falls to zero.

The other branch of the Big Muff Pi tone control is a LPF with a 0.01μF capacitor and a 39K resistor (see a schematic from the layouts section of gauss-markov.net). The SPICE graph of the gain measured in decibels appears in Figure 8. This is computed comparably to the HPF case; see Figure 7. In the Big Muff Pi circuit, these two passive filters are blended by the 10K tone pot.

9 Cut-Off Frequency

The *cut-off frequency* of a filter is defined as "the frequency where the signal being filtered is at half power." Half power is when

$$\frac{\|V_{out}\|^2}{\|V_{in}\|^2} = \frac{1}{2}$$

or

$$\frac{\|V_{out}\|}{\|V_{in}\|} = \frac{1}{\sqrt{2}} \approx 0.71$$

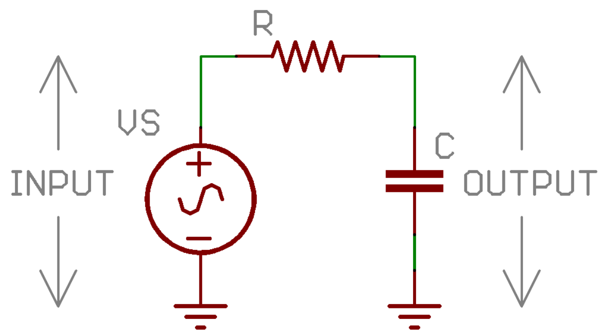

Figure 6: RC Low Pass Filter Schematic

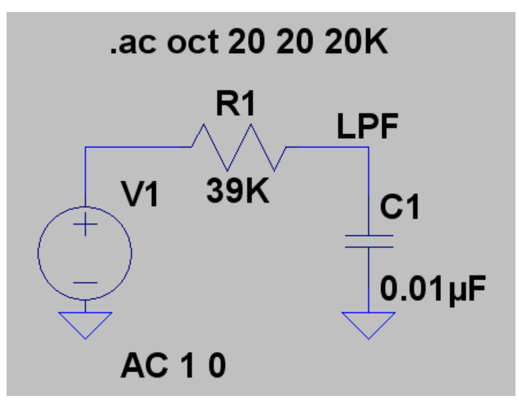

Figure 7: SPICE Low Pass Filter AC Analysis

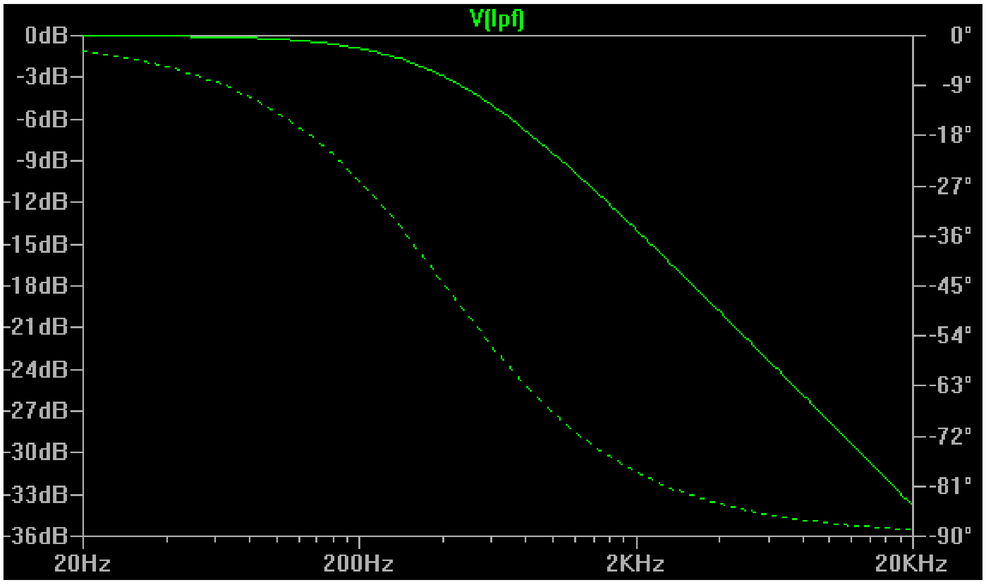

Figure 8: SPICE Low Pass Filter AC Analysis

or when the amplitude of the output signal is about 71% of the amplitude of the input signal.

To calculate the cut-off frequency for the passive HPF, we set the gain to $1/\sqrt{2}$ in equation (1) above:

$$\frac{1}{\sqrt{2}} = \frac{2\pi f_c RC}{\sqrt{1+(2\pi f_c RC)^2}}$$

and we solve for f_c to obtain

$$f_c = \frac{1}{2\pi RC}$$

where f_c is measured in hertz. For the passive LPF,

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{1+(2\pi f_c RC)^2}}$$

which has the same solution for f_c. As a reflection of this coincidence of cut-off frequencies, note that the cut-off frequency is also the frequency where resistance equals reactance in both filters:

$$R = \frac{1}{2\pi f_c C}$$

For a numerical example, let's return to the HPF in the Big Muff Pi. Its cut-off frequency is

$$f_c = \frac{1}{2\pi\left(39\times 10^3\right)\left(0.004\times 10^{-6}\right)} \approx 1020.22399418\text{Hz}$$

Given our previous calculations for the HPF, this illustrates a rule of thumb: the the cutoff frequency occurs at

$$20\log_{10}\left(\frac{1}{\sqrt{2}}\right) \approx -3\text{dB}$$

from the input level.

Looking at either graph, one sees that the cut-off frequency is not where output is literally cut off. Another rule of thumb is that the output from a passive filter drops 6dB for every octave beyond the cutoff frequency. If we calculate the drop off in the LPF from one octave beyond the cut-off frequency to five octaves beyond, we get

$$\begin{aligned}
&20\log_{10}\left(\frac{1}{\sqrt{1+(2\pi(2f_c)RC)^2}}\right) - 20\log_{10}\left(\frac{1}{\sqrt{1+(2\pi\left(2^5 f_c\right)RC)^2}}\right) \\
&= 20\log_{10}\left(\frac{1}{\sqrt{1+2^2}}\right) - 20\log_{10}\left(\frac{1}{\sqrt{1+\left(2^5\right)^2}}\right) \\
&= 10\log(205) \\
&\approx 23.12
\end{aligned}$$

which corresponds to an average $23.12/4 = 5.8$ decibels per octave. A comparable calculation for the HPF yields the same answer. So the first octave is not a 6dB drop but the rest are pretty close. It is a nice number because it corresponds roughly to halving the volume for each octave.

10 A Final Note

Note that these calculations assume that the AC input is not affected by the filter and the destination of the output does not affect the filter. We assumed an ideal voltage source, which supplies a specified AC voltage and, therefore, also supplies whatever current is required to accompany this voltage. We also assumed an ideal output with no connection between ground and the junction labelled "output." In application, such conditions often do not hold.

In general, components surrounding this filter in a larger circuit will alter the behaviour described above. As an example, look at the tone section of a Big Muff Pi again (see a schematic from the layouts section of gaussmarkov.net). The output of each filter encounters resistance at the 10K tone pot that blends the filters. That resistance will "load down" the filters.

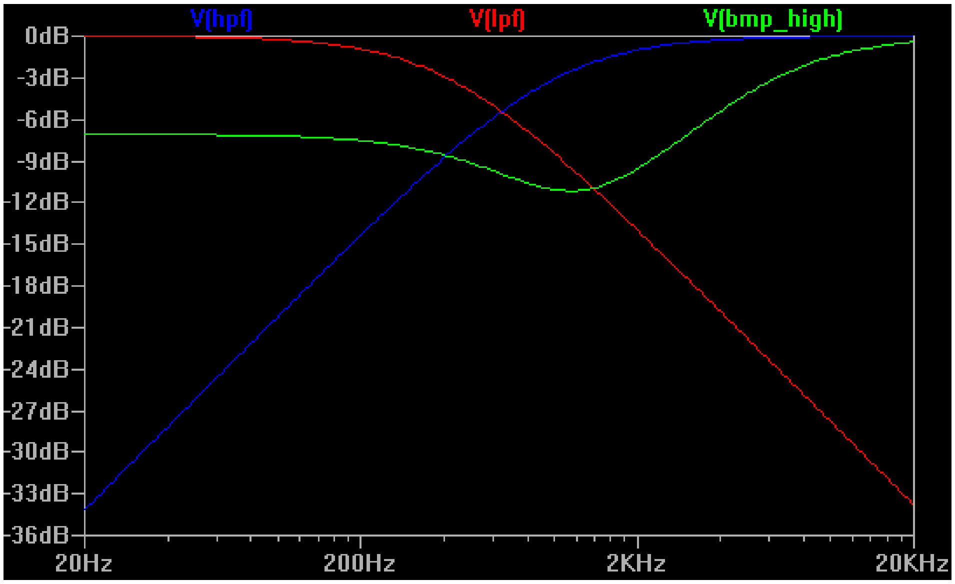

Figure 9: BMP Tone Clockwise

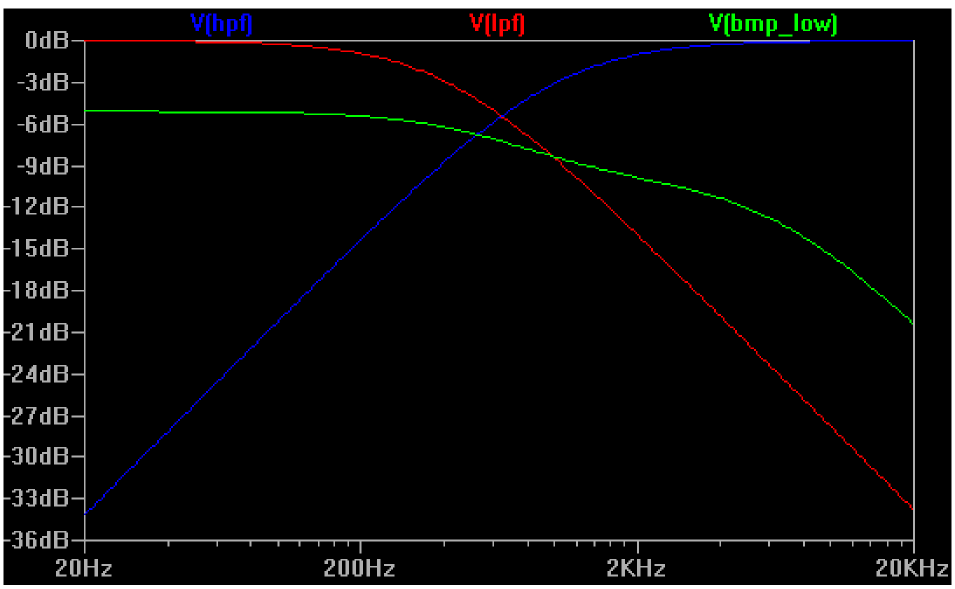

Figure 10: BMP Tone Counter-Clockwise

Figures 9 and 10 show the gains of the two passive filters and the BMP tone section with the pot set clockwise for high pass filtering and counter-clockwise for low pass filtering. Notice, for example, that the lowest frequencies are attenuated quite a bit in both cases. The figures also show that the two extremes do not correspond to passive high and low pass filters or simple weighted averages of the HPF and LPF.