Negative Feedback with Capacitors

http://gaussmarkov.net

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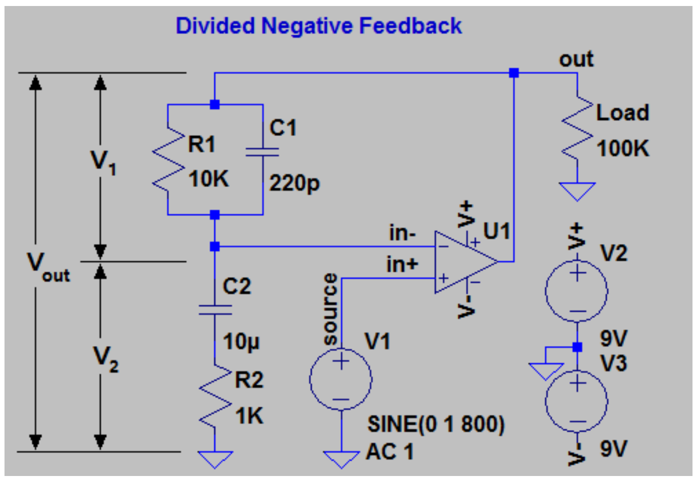


The web page Op-Amps 4: Negative Feedback, describes several characteristics of the gain of a circuit like this one. This document provides the mathematical derivations of these characteristics.

1 Subcircuits

First, each resistor-capacitor pair is described separately. This gives expressions for V_1 and V_2, the voltages across each pair as indicated in the schematic above. Second, these voltages are combined to find the output and gain of the op-amp.

In the section that follows this one, the expression for gain is summarized in terms of its maximum, symmetry, and corner frequencies.

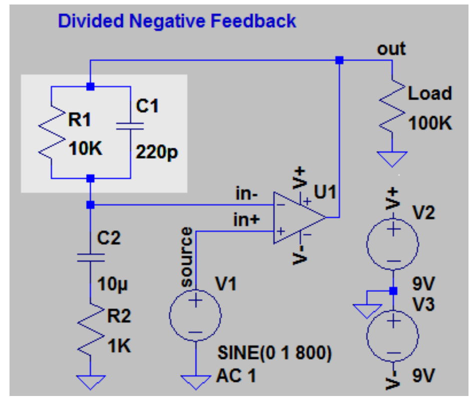
1.1 Resistor and Capacitor in Parallel

A resistance R_1 and capacitance C_1 in parallel have the voltage V_1 across their leads so that

$$V_1 = I_R R_1 \qquad \text{by Ohm's law for resistors,}$$

$$I_C = C_1 \frac{dV_1}{dt} \qquad \text{by the capacitor law,}$$

$$I_1 = I_R + I_C \qquad \text{by Kirchoff's current law,}$$



where I_R is the current through the resistor, I_C is the current through the capacitor, and I_1 is the current through the parallel combination. These combine to give the relationship between current I_1 and voltage V_1:

$$I_1 R_1 = V_1 + \tau_1 \frac{dV_1}{dt} \tag{1}$$

where $\tau_1 = R_1 C_1$.

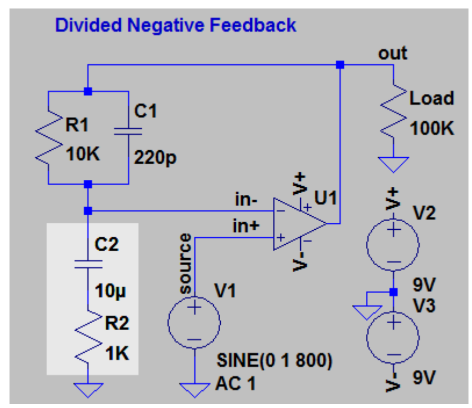
1.2 Resistor and Capacitor in Series

A resistance R_2 and a capacitance C_2 in series have the same current I_2 so that

$$V_R = I_2 R_2 \qquad \text{by Ohm's law for resistors,}$$

$$I_2 = C_2 \frac{dV_C}{dt} \qquad \text{by the capacitor law,} \tag{2}$$

$$V_2 = V_C + V_R \qquad \text{by Kirchoff's voltage law,}$$

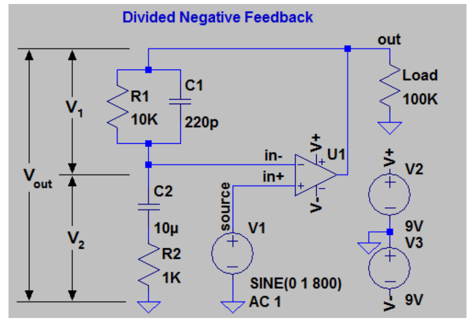


where V_R is the voltage across the resistor, V_C is the voltage across the capacitor, and V_2 is the voltage across the pair. These combine to give V_2 in terms of the voltage V_C

$$V_2 = V_C + \tau_2 \frac{dV_C}{dt} \tag{3}$$

where $\tau_2 = R_2 C_2$.

1.3 Op-Amp with Divided Negative Feedback



Consider an op-amp with split supply and divided negative feedback where R_1 and C_2 are in parallel between the output and the inverting input and R_2 and C_2 are in series from the inverting input to ground. Then, in addition to the relationships above,[1]

$V_2 = V_{\text{in}-} \approx V_{\text{in}+} = V_{\text{source}}$	op-amp with feedback
$I_1 = I_2$	Kirchoff's current law
$V_{\text{out}} = V_1 + V_2$	Kirchoff's voltage law

Now we can describe a solution strategy.

1. V_{source} determines V_2.
2. V_2 will determine I_2 through the combined behaviour of R2 and C2.
3. I_2 equals I_1.
4. I_1 will determine V_1 through the combined behaviour of R1 and C1.
5. Finally, V_1 and V_2 sum to give V_{out}.

Steps 2 and 4 will use the fact that the stationary solution to the differential equation

$$a\cos\omega t + b\sin\omega t = y(t) + c\frac{dy(t)}{dt}$$

is

$$y = \frac{1}{1+\omega^2c^2}\big((a - bc\omega)\cos t\omega + (b + ac\omega)\sin t\omega\big). \tag{4}$$

2 Circuit Output

Step 1: Following the standard approach, let the input signal be a simple sine wave so that

$$V_2 = V_{\text{in}+} = \sin\omega t \tag{5}$$

where ω is an abbreviation for $2\pi f$ where f is the frequency.

[1] When V_{out} is not clipped by the power rails, then

$$V_{\text{out}} = 200{,}000 \times (V_{\text{in}+} - V_{\text{in}-})$$

implies that

$$V_{\text{in}+} - V_{\text{in}-} = \frac{V_{\text{out}}}{200{,}000} \approx 0.$$

Step 2: Then (3) gives the differential equation for V_C

$$\sin\omega t = V_C + \tau_2 \frac{dV_C}{dt}$$

which has the stationary solution (using equation 4)

$$V_C = \frac{1}{1+\omega^2\tau_2^2}\left(\sin\omega t - \omega\tau_2\cos\omega t\right).$$

Therefore, using (2), the current through this capacitor is

$$\begin{aligned} I_2 &= C_2\frac{dV_C}{dt} \\ &= \frac{C_2\omega}{1+\omega^2\tau_2^2}\left(\cos\omega t + \omega\tau_2\sin\omega t\right). \end{aligned}$$

Steps 3 and 4: Kirchoff's current law also implies that $I_2 = I_1$ or, using (1),

$$\frac{\rho\tau_2\omega}{1+\omega^2\tau_2^2}\left(\cos\omega t + \omega\tau_2\sin\omega t\right) = V_1 + \tau_1\frac{dV_1}{dt}$$

where $\rho = R_1/R_2$. Again using (4), the solution to this differential equation is

$$V_1 = \frac{\rho\tau_2\omega}{\left(1+\omega^2\tau_1^2\right)\left(1+\omega^2\tau_2^2\right)}\Big(\left(1-\omega^2\tau_1\tau_2\right)\cos t\omega + (\tau_1+\tau_2)\omega\sin t\omega\Big). \tag{6}$$

Step 5: These solutions for V_1 and V_2, equations (6) and (5), allow us to solve for V_{out}:

$$\begin{aligned} V_{\text{out}} &= V_1 + V_2 \\ &= \frac{\rho\tau_2\omega}{\left(1+\omega^2\tau_1^2\right)\left(1+\omega^2\tau_2^2\right)}\Big(\left(1-\omega^2\tau_1\tau_2\right)\cos t\omega + (\tau_1+\tau_2)\omega\sin t\omega\Big) + \sin\omega t \end{aligned}$$

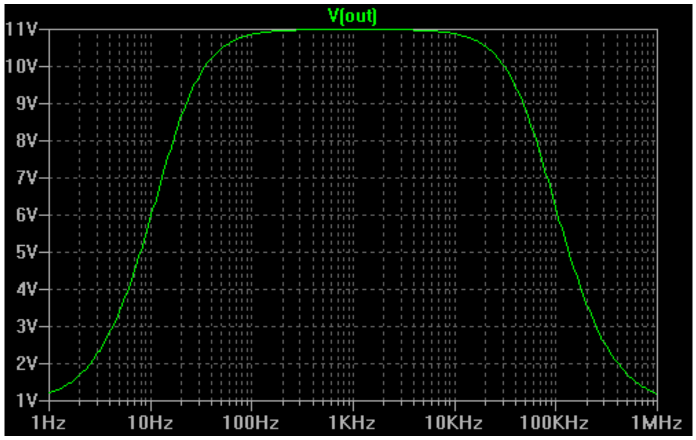
Because we specified an input with an amplitude equal to one, the squared gain of the circuit is the squared amplitude of V_{out}. The squared amplitude of V_{out} equals the sum of the squared coefficients of the sin and cos terms.

$$(\text{gain})^2 = \left(\frac{\rho\tau_2\omega\left(1-\omega^2\tau_1\tau_2\right)}{\left(1+\omega^2\tau_1^2\right)\left(1+\omega^2\tau_2^2\right)}\right)^2 + \left(1 + \frac{(\tau_1+\tau_2)\rho\tau_2\omega^2}{\left(1+\omega^2\tau_1^2\right)\left(1+\omega^2\tau_2^2\right)}\right)^2 \tag{7}$$

$$= 1 + \frac{\left(2(\tau_1+\tau_2)+\rho\tau_2\right)\rho\tau_2\omega^2}{\left(1+\omega^2\tau_1^2\right)\left(1+\omega^2\tau_2^2\right)}. \tag{8}$$

Like the special case of a resistive voltage divider, this gain is greater than one.

As a function of frequency $f = \omega/(2\pi)$, the square root of (8) is the gain plotted by LTSpice in this graphic, generated with the component values in the schematics above:



There are several things about this plot that hold for every gain-frequency plot no matter what the component values are:

1. one maximum
2. unity gain at the lowest and highest frequencies
3. symmetry in log-frequency around the maximum

In addition, for component values typically found in stompbox circuits,

4. the maximum gains is approximately equal to the no-capacitor gain and
5. lower and upper corner frequencies summarize this shape.

The next section verifies these properties.

3 Properties of the Gain Function

3.1 Extreme Values

Consider the part of gain that depends on ω:

$$\frac{\omega^2}{\left(1+\omega^2\tau_1^2\right)\left(1+\omega^2\tau_2^2\right)} = \frac{1}{\tau_1\tau_2\left(\omega^2\tau_1\tau_2 + \left(\omega^2\tau_1\tau_2\right)^{-1} + \frac{\tau_2}{\tau_1} + \frac{\tau_1}{\tau_2}\right)}.$$

Because the function

$$x + x^{-1}, \qquad x > 0$$

has a unique global minimum equal to 2 at $x = 1$, this part of the gain function has a unique global maximum over values of ω at

$$\omega_*^2\tau_1\tau_2 = 1 \qquad \Longleftrightarrow \qquad \omega_* = \sqrt{\frac{1}{\tau_1\tau_2}}$$

so that the maximum squared gain is

$$1 + \frac{\left(2(\tau_1+\tau_2)+\rho\tau_2\right)\rho\tau_2}{(\tau_1+\tau_2)^2} = \left(1+\rho\frac{\tau_2}{\tau_1+\tau_2}\right)^2$$

at the frequency

$$f_* = \frac{1}{2\pi}\sqrt{\frac{1}{\tau_1\tau_2}}.$$

This maximum is slightly modified from the pure resistive case where squared gain is $\left(1+\rho\right)^2$ for all frequencies.

The minima of the gain profile occur at zero and infinite frequencies where the gain is unity because

$$\lim_{x\to 0^+}\frac{1}{x+x^{-1}} = \lim_{x\to\infty}\frac{1}{x+x^{-1}} = 0.$$

3.2 Symmetry

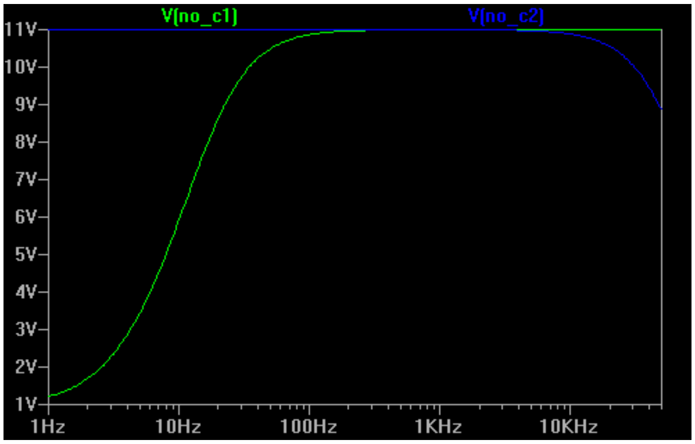
Written in terms of log-frequencies, the gain function is symmetric around its maximum because

$$ax+(ax)^{-1} = 10^{z-b}+10^{-(z-b)} = 10^{|z-b|}+10^{-|z-b|}$$

where $x = 10^z$ and $a = 10^{-b}$.

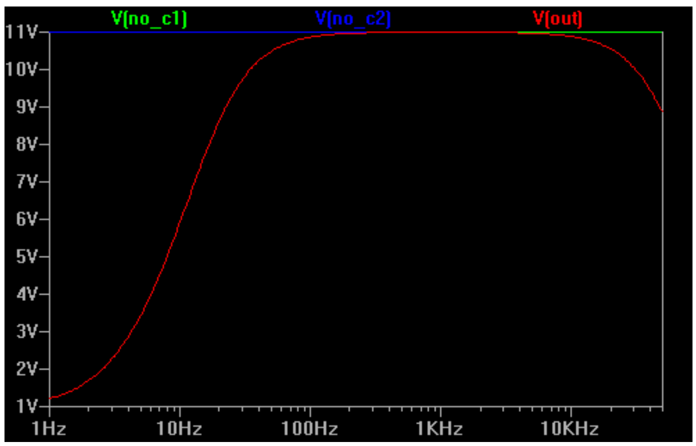
3.3 Corner Frequencies

The corner frequencies are easier to study when you notice that they are very nearly the corner frequencies for two special cases: when one or the other capacitor is not present. Here are two plots to show the idea. In the first one, the green line shows the gain function when there is no C1 capacitor and the blue line shows the gain functions when there is no C2.



Without C1, the output looks like something like a high pass filter. The difference is that the highs are amplified rather than the lows being attenuated. Without C2, we get more lows than highs. In both cases, there is range of frequencies where the capacitor that is present ceases to have an appreciable effect. For those frequencies the gain equals the gain for the resistive voltage divider.

The second plot adds a red line for the gain with both C1 and C2. The red line lies almost exactly on the lower value of the two other lines. For the component values that we are using, there are no frequencies where both capacitors are effective. This is the reason that the red line agrees with parts of the green and blue lines.



As a result, we can use corner frequencies of the blue and green lines to accurately approximate the corner frequencies of the red line. We will derive the two special cases as limits of the general case.

When $C_1 \to 0$, then it is as though there is no C1 cap. When $C_2 \to \infty$, then it is as though there is no C2 cap. These two cases correspond to $\tau_1 \to 0$ and $\tau_2 \to 0$, respectively. The squared gain limits are

$$\lim_{\tau_1 \to 0} 1 + \frac{\left(2(\tau_1+\tau_2)+\rho\tau_2\right)\rho\tau_2\omega^2}{\left(1+\omega^2\tau_1^2\right)\left(1+\omega^2\tau_2^2\right)} = 1 + \frac{\rho\left(\rho+2\right)}{1+1/(\omega\tau_2)^2}$$

and

$$\lim_{\tau_2 \to \infty} \left(1 + \frac{\left(2(\tau_1+\tau_2)+\rho\tau_2\right)\rho\tau_2}{\left(1/\omega+\tau_1^2\omega\right)\left(1/\omega+\tau_2^2\omega\right)}\right) = 1 + \frac{\rho\left(\rho+2\right)}{1+(\omega\tau_1)^2}.$$

The maximum squared gain limits are

$$\lim_{\tau_1 \to 0} \left(1+\rho\frac{\tau_2}{\tau_1+\tau_2}\right)^2 = \left(1+\rho\right)^2$$

and

$$\lim_{\tau_2 \to \infty} \left(1+\rho\frac{\tau_2}{\tau_1+\tau_2}\right)^2 = \left(1+\rho\right)^2.$$

In both cases, the maximum gain simplifies to the simpler, no-capacitor case.

The corner frequency for $\tau_1 = 0$ is given by the frequency that solves

$$\frac{1 + \frac{\rho(\rho+2)}{1+1/(\omega^2\tau_2^2)}}{(1+\rho)^2} = \frac{1}{2}$$

Naturally, a solution exists only if the maximum gain is greater than 2 (or $\rho > 1$). Using $\omega = 2\pi f$,

$$f_2 = \frac{1}{2\pi\tau_2}\sqrt{1 - \frac{2}{(1+\rho)^2}}$$

Similarly, for the $\tau_2 \to \infty$ case,

$$\frac{1 + \frac{\rho(\rho+2)}{1+(\omega\tau_1)^2}}{1+\rho^2} = \frac{1}{2}$$

gives

$$f_1 = \frac{1}{2\pi\tau_1}\frac{1}{\sqrt{1 - \frac{2}{(1+\rho)^2}}}.$$

The passband is the range of frequencies between these two corner frequencies. As $\rho \to 1^+$, $f_2 \to 0$ and $f_1 \to \infty$ and the passband approaches all frequencies. For $\rho > 5$, the factor depending on ρ can be ignored so that $f_2 \approx 1/(2\pi\tau_2)$ and $f_1 \approx 1/(2\pi\tau_1)$. This corresponds to $14dB$ gain.

These approximations to the corner frequencies work well as long as there is an interval in which both capacitors have little effect. Such an interval contains frequencies for which gain is approximately the resistive divider gain:

$$(1+\rho)^2 \approx 1 + \frac{\rho(\rho+2)}{1+1/(\omega\tau_2)^2}, \quad 1 + \frac{\rho(\rho+2)}{1+(\omega\tau_1)^2}.$$

We can solve these approximations as equalities to find values for the angular frequency ω that attain a gain of $\beta(1+\rho)^2$:

$$\begin{aligned}\omega_2^2 &= \frac{\beta - (1+\rho)^{-2}}{\tau_2^2(1-\beta)}, \\ \omega_1^2 &= \frac{1}{\tau_1^2}\frac{(1-\beta)}{\beta - (1+\rho)^{-2}}.\end{aligned}$$

We require $\omega_2 \leq \omega_1$ which implies

$$\frac{\tau_1}{\tau_2} \leq \frac{1-\beta}{\beta - (1+\rho)^{-2}}.$$

Taking $\beta \approx 1$, this requires

$$\left(1 - (1+\rho)^{-2}\right)\frac{\tau_1}{\tau_2} \leq 1 - \beta$$

or

$$\left(1-(1+\rho)^{-2}\right)\frac{f_2}{f_1} \leq 1-\beta.$$